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INVESTIGATION OF A CLASS OF TWO-DIMENSIONAL CONJUGATE INTEGRAL EQUATION WITH FIXED SUPER-SINGULAR KERNELS

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Abstract. In this paper, two-dimensional linear conjugate Volterra integral equations containing super-singularities in the kernels are considered. The existence of a unique solution in a certain function class is established. Formulas representing the solution are given.

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Keywords: conjugate Volterra integral equation, super-singularity in kernel, two-dimensional integral equation

1. Introduction

Let D denote the rectangle $D = \{a < x < a_0,\ b_0 < y < b\}$ and introduce the sets $\Gamma_1 = \{a < x < a_0,\ y = b\}$ and $\Gamma_2 = \{x = a,\ b_0 < y < b\}$. In D, we consider the two-dimensional integral equation

$$u(x,y) + \lambda \int_a^x \frac{u(t,y)}{(t-a)^\alpha} dt - \mu \int_y^b \frac{u(x,s)}{(b-s)^\beta} ds + \delta \int_a^x \frac{dt}{(t-a)^\alpha} \int_y^b \frac{u(t,s)}{(b-s)^\beta} ds = f(x,y), \quad (1.1)$$

and the integral equation conjugate to equation (1.1)

$$T^{\alpha,\beta}_{\lambda,\mu}(v) \equiv v(x,y) + \frac{\lambda}{(x-a)^\alpha} \int_x^{a_0} v(t,y)dt - \frac{\mu}{(b-y)^\beta} \int_{b_0}^y v(x,s)ds + \frac{\delta}{(x-a)^\alpha (b-y)^\beta} \int_x^{a_0} dt \int_{b_0}^y v(t,s)ds = g(x,y), \quad (1.2)$$

where $\alpha > 1$, $\beta > 1$, $\{\lambda,\mu,\delta\} \subset \mathbb{R}$, $f(x,y)$, $g(x,y)$ are the given functions and $u(x,y)$, $v(x,y)$, are the unknown functions.

By the study of the solutions of the integral equations (1.1) and (1.2), the problem can be reduced to the determination of the continuous solution of a hyperbolic equation with two super-singular lines and its conjugate equation in D.

We seek a solution of equation (1.1) in the class of functions $u(x,y) \in C(\overline{D})$ that vanish on the singular lines Γ_1 and Γ_2. Moreover, we will assume that the unknown function $u(x,y)$ vanishes as $x \to a$ by an order higher than $\alpha - 1$, and it vanishes as $y \to b$ by an order higher than $\beta - 1$.

We note that the one-dimensional integral equations of types (1.1) and (1.2) are studied in [2,4–6]. Two-dimensional , three-dimensional and some many-dimensional Volterra type integral equations of type (1.1) are studied in [2,3,7–9]. One-dimensional singular integral equations with Cauchy kernels are considered in [1].

2. THE CASE WHERE $\delta = -\lambda\mu$

In this case, the integral equation (1.2) can be represented in the following form:

$$v(x,y) - \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} v(x,s)ds + \frac{\lambda}{(x-a)^\alpha}\int_{x}^{a_0}[v(t,y) - \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} v(t,s)ds]dt = g(x,y).$$

If we introduce a new unknown function

$$W(x,y) = v(x,y) - \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} v(x,s)ds, \tag{2.1}$$

we arrive to a one-dimensional conjugate Volterra type integral equation

$$W(x,y) + \frac{\lambda}{(x-a)^\alpha}\int_{x}^{a_0} W(t,y)dt = g(x,y). \tag{2.2}$$

In the case where $a < x < a_0$, according to [3], the integral equation (2.2) has a unique solution which is given by the formula

$$W(x,y) = g(x,y) - \frac{\lambda}{(x-a)^\alpha}\int_{x}^{a_0} \exp[\lambda(\omega_a^\alpha(t) - \omega_a^\alpha(x))]g(t,y)dt, \tag{2.3}$$

where $\omega_a^\alpha(x) = [(\alpha-1)(x-a)^{\alpha-1}]^{-1}$.

Analogously, the solution of the integral equation (2.1), for $b_0 < y < b$, is given by the formula

$$v(x,y) = W(x,y) + \frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y) - \omega_b^\beta(s))]W(t,y)ds, \tag{2.4}$$

where $\omega_b^\beta(y) = [(\beta-1)(b-y)^{\beta-1}]^{-1}$.

By substituting the value of $W(x,y)$ into (2.4), we obtain a general solution of the integral equation (1.2) in the form

$$v(x,y) = g(x,y) - \frac{\lambda}{(x-a)^\alpha}\int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t)-\omega_a^\alpha(x))]g(t,y)dt$$
$$+\frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y)-\omega_b^\beta(s))]g(x,s)ds$$
$$-\frac{\lambda\mu}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t)-\omega_a^\alpha(x))]dt$$
$$\times\int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y)-\omega_b^\beta(s))]g(t,s)ds \equiv (T_{\lambda,\mu}^{\alpha,\beta})^{-1}g. \quad (2.5)$$

We thus obtain the following

Theorem 1. *Assume that in equation (1.2) the parameters are related by the equality $\delta = -\lambda\mu$, and $g(x,y) \in C(D)$. Then the non-homogeneous integral equation (1.2) has a unique solution in class $C(D)$, which is given by formula (2.5).*

3. The case where $\delta \neq -\lambda\mu$

In this case, the integral equation (1.2) can be represented in the following form:

$$T_{\lambda,\mu}^{\alpha,\beta}(v) \equiv v(x,y) + \frac{\lambda}{(x-a)^\alpha}\int_x^{a_0} v(t,y)dt$$
$$-\frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} v(x,s)ds - \frac{\lambda\mu}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} dt\int_{b_0}^{y} v(t,s)ds$$
$$= g(x,y) - \frac{\delta_1}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} dt\int_{b_0}^{y} v(t,s)ds.$$

Let us introduce the function

$$g_1(x,y) = g(x,y) - \frac{\delta_1}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} dt\int_{b_0}^{y} v(t,s)ds,$$

where $\delta_1 = \delta + \lambda\mu$. Clearly, $g_1(x,y) \in C(D)$. Then the solution of the integral equation $T_{\lambda,\mu}^{\alpha,\beta}(v) = g_1(x,y)$ is as follows:

$$v(x,y) = g_1(x,y) - \frac{\lambda}{(x-a)^\alpha}\int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t)-\omega_a^\alpha(x))]g_1(t,y)dt$$
$$+\frac{\mu}{(b-y)^\beta}\int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y)-\omega_b^\beta(s))]g_1(x,s)ds$$
$$-\frac{\lambda\mu}{(x-a)^\alpha(b-y)^\beta}\int_x^{a_0} \exp[\lambda(\omega_a^\alpha(t)-\omega_a^\alpha(x))]dt$$
$$\times\int_{b_0}^{y} \exp[\mu(\omega_b^\beta(y)-\omega_b^\beta(s))]g_1(t,s)ds \equiv (T_{\lambda,\mu}^{\alpha,\beta})^{-1}g_1(x,y). \quad (3.1)$$

In formula (3.1), by substituting the value of $g_1(x,y)$ and rearranging the appropriate terms, we arrive to the solution of the integral equation

$$\Phi(x,y)+\frac{\delta_1}{(x-a)^{\alpha}(b-y)^{\beta}}\int_x^{a_0}dt\int_{b_0}^{y}\Phi(t,s)ds=E^{\alpha,\beta}_{\lambda,\mu}[g(x,y)], \tag{3.2}$$

where

$$\begin{aligned}E^{\alpha,\beta}_{\lambda,\mu}[g(x,y)]&=\exp[\lambda\omega^{\alpha}_{a}(x)-\mu\omega^{\beta}_{b}(y)](T^{\alpha,\beta}_{\lambda,\mu})^{-1}g(x,y)\\&=\exp[\lambda\omega^{\alpha}_{a}(x)-\mu\omega^{\beta}_{b}(y)]g(x,y)\\&-\frac{\lambda}{(x-a)^{\alpha}}\exp[-\mu\omega^{\beta}_{b}(y)]\int_x^{a_0}\exp[\lambda\omega^{\alpha}_{a}(t)]g(t,y)dt\\&+\frac{\mu}{(b-y)^{\beta}}\exp[\lambda\omega^{\alpha}_{a}(x)]\int_{b_0}^{y}\exp[-\mu\omega^{\beta}_{b}(s)]g(x,s)ds\\&-\frac{\lambda\mu}{(x-a)^{\alpha}(b-y)^{\beta}}\int_x^{a_0}\exp[\lambda\omega^{\alpha}_{a}(t)]dt\int_{b_0}^{y}\exp[-\mu\omega^{\beta}_{b}(s)]g(t,s)ds\end{aligned} \tag{3.3}$$

and

$$\Phi(x,y)=\exp[\lambda\omega^{\alpha}_{a}(x)-\mu\omega^{\alpha}_{a}(y)]v(x,y).$$

4. Representation of a solution by functional series of $\exp(-\omega^{\alpha}_{a}(x))$

We seek a solution for integral equation (3.2) in the class of functions that can be represented in the form

$$\Phi(x,y)=\sum_{n=1}^{\infty}(\exp(-\omega^{\alpha}_{a}(x)))^{n}\Phi_n(y)(x-a)^{-\alpha}, \tag{4.1}$$

where $\Phi_n(y)$ are unknown functions.

We assume that function $g(x,y)$ admits representation in the form

$$g(x,y)=\exp[-\lambda\omega^{\alpha}_{a}(x)+\mu\omega^{\beta}_{b}(y)]\sum_{n=1}^{\infty}[\exp(-\omega^{\alpha}_{a}(x))^{n}](x-a)^{-\alpha}g_n(y), \tag{4.2}$$

where $g_n(y)$ are known functions. Moreover, assume that the series (4.2) converges absolutely and uniformly. By substituting this value $g(x,y)$ into (3.3), we have

$$\begin{aligned}E^{\alpha,\beta}_{\lambda,\mu}[g(x,y)]&=\sum_{n=1}^{\infty}[\exp(-\omega^{\alpha}_{a}(x))]^{n}(x-a)^{-\alpha}\\&\times\left(\frac{n+\lambda}{n}\right)\left[g_n(y)+\mu(b-y)^{-\mu}\int_{b_0}^{y}g_n(s)ds\right]\\&-(x-a)^{-\alpha}\lambda\left[\sum_{n=1}^{\infty}n^{-1}\exp(-\omega^{\alpha}_{a}(a_0))^{n}\left(g_n(y)+\mu(b-y)^{-\mu}\int_{b_0}^{y}g_n(s)ds\right)\right].\end{aligned}$$

By substituting the values of $\Phi(x,y)$ and $E^{\alpha,\beta}_{\lambda,\mu}[g(x,y)]$ into the integral equation (3.2) and equating the coefficients at $[\exp(-\omega^{\alpha}_{a}(x))]^k$, $k=0,1,2,\dots,$ we obtain the following relations between the functions $\Phi_n(y)$ and $g_n(y)$, $n=0,1,2,\dots$:

$$\delta_1(b-y)^{-\beta}\sum_{n=1}^{\infty} n^{-1}[\exp(-\omega^{\alpha}_{a}(a_0))]^n \int_{b_0}^{y}\Phi_n(s)ds$$
$$= -\sum_{n=1}^{\infty} n^{-1}[\exp(-\omega^{\alpha}_{a}(a_0))]^n \lambda[g_n(y)+(b-y)^{-\beta}\mu\int_{b_0}^{y} g_n(s)ds] \quad (4.3)$$

and

$$\Phi_n(y)-\frac{\delta_1}{n(b-y)^{\beta}}\int_{b_0}^{y}\Phi_n(s)ds = \left(\frac{n+\lambda}{n}\right)g_n(y)+\frac{\mu(n+\lambda)}{n(b-y)^{\mu}}\int_{b_0}^{y} g_n(s)ds. \quad (4.4)$$

According to [3], if the system of integral equation (4.4) has a solution, then it can be represented in the form

$$\Phi_n(y)=\frac{n+\lambda}{n}\Bigg[g_n(y)$$
$$+\left(\frac{\mu n-\delta_1}{n}\right)\frac{1}{(b-y)^{\beta}}\int_{b_0}^{y}\exp\left[\frac{\delta_1}{n}(\omega^{\beta}_{b}(s)-\omega^{\beta}_{b}(y))\right]g_n(s)ds\Bigg], \quad (4.5)$$

where $n=0,1,2,\dots,$ $\omega^{\beta}_{b}(y)=[(\beta-1)(b-y)^{\beta-1}]^{-1}$.

Furthermore, it follows from equality (4.3) that

$$\delta_1(b-y)^{-\beta}\int_{b_0}^{y}\Phi_n(s)ds=-\lambda[g_n(y)+\mu(b-y)^{-\beta}\int_{b_0}^{y} g_n(s)ds] \quad (4.6)$$

for $n=0,1,2,\dots$. From expression (4.6), by substituting the values of $\Phi_n(s)$ according to formula (4.5), we obtain the equality

$$\left(\frac{n+\lambda}{n}\right)\Bigg[\frac{\mu n}{(b-y)^{\beta}}\int_{b_0}^{y} g_n(s)ds$$
$$-\frac{(\mu n-\delta_1)}{(b-y)^{\beta}}\int_{b_0}^{y}\exp\left(\frac{\delta_1}{n}(\omega^{\beta}_{b}(s)-\omega^{\beta}_{b}(y))\right)g_n(s)ds\Bigg]$$
$$=-\lambda\left[g_n(y)+\frac{\mu}{(b-y)^{\beta}}\int_{b_0}^{y} g_n(s)ds\right],\quad n=0,1,2,3\dots. \quad (4.7)$$

From formula (4.1), by substituting the value $\Phi_n(y)$ from equality (4.5), where $\Phi(x,y)=v(x,y)\exp[\lambda\omega^{\alpha}_{a}(x)-\mu\omega^{\beta}_{b}(y)]$, we find

$$v(x,y) = \exp[-\lambda\omega_a^\alpha(x) + \mu\omega_b^\beta(y)] \sum_{n=1}^{\infty} \frac{\exp(-\omega_a^\alpha(x))^n}{(x-a)^\alpha} \left(\frac{n+\lambda}{n}\right) \\ \times \left[g_n(y) + \frac{\mu n - \delta_1}{n(b-y)^\beta} \int_{b_0}^{y} (\exp(\omega_b^\beta(s) - \omega_b^\beta(y)))^{\frac{\delta_1}{n}} g_n(s) ds \right]. \quad (4.8)$$

Thus, we arrive at the following conclusion.

Theorem 2. *Assume that in the integral equation (1.2) $\delta \neq -\lambda\mu$, and that the function $g(x,y)$ is represented by series (4.2), which converges absolutely and uniformly. Then the integral equation (1.2) has a solution in the class of functions $v(x,y)$ that are representable in the form*

$$v(x,y) = \exp[-\lambda\omega_a^\alpha(x) + \mu\omega_b^\beta(y)] \sum_{n=1}^{\infty} \frac{\Phi_n(y)}{(x-a)^\alpha \exp(\omega_a^\alpha(x))^n}.$$

Moreover, if the functions $g_k(y)$, $k = 1,2,\ldots$, in (4.2) satisfy the infinite system of solvability conditions (4.7), then that solution is unique and can be represented by by formula (4.8).

Remark 1. In the case where $\delta \neq -\lambda\mu$, the solution of the integral equation (3.2) could be sought in the class of functions that are representable by a functional series of $\exp(-\omega_b^\beta(y))$, i. e.,

$$\Phi(x,y) = \sum_{n=1}^{\infty} \exp(-n\omega_b^\beta(y))(b-y)^{-\beta} W_n(x)$$

where $W_n(x)$ are unknown functions. Then one assumes that the function $g(x,y)$ is represented in the form

$$g(x,y) = \exp[-\lambda\omega_a^\alpha(x) + \mu\omega_b^\beta(y)] \sum_{n=1}^{\infty} \frac{\exp(-n\omega_b^\beta(y))}{(b-y)^\beta} g_n(x).$$

By modifying suitably the argument above, in that case, one can also obtain a statement similar to Theorem 2.

5. Remarks on a non-model integral equation

In the domain D, we consider the two-dimensional integral equation

$$u(x,y) + \int_a^x \frac{A(t)u(t,y)}{(t-a)^\alpha} dt - \int_y^b \frac{B(s)u(x,s)}{(b-s)^\beta} ds \\ + \int_a^x \frac{dt}{(t-a)^\alpha} \int_y^b \frac{c(t,s)u(t,s)}{(b-s)^\beta} ds = f(x,y), \quad (5.1)$$

and its conjugate equation

$$v(x,y)+\frac{A(x)}{(x-a)^{\alpha}}\int_{x}^{a_0} v(t,y)dt-\frac{B(y)}{(b-y)^{\beta}}\int_{b_0}^{y} v(x,s)ds$$
$$+\frac{c(x,y)}{(x-a)^{\alpha}(b-y)^{\beta}}\int_{x}^{a_0} dt\int_{b_0}^{y} v(t,s)ds=g(x,y). \quad (5.2)$$

Integral equations of form (5.1) are studied in [8].

Remark 2. One can find a solution of the integral equation (5.2) if $c(x,y)\equiv -A(x)B(y)$. In that case, as is shown in [6], the question is reduced to finding a solution of two split systems of one-dimensional conjugate integral equations of type (5.2).

Remark 3. In the case where $c(x,y)\not\equiv -A(x)B(y)$, the problem of finding solution for integral equation (5.2) is reduced to the problem of the determination of a solution of the integral equation

$$v(x,y)$$
$$+\frac{c_1(x,y)}{(x-a)^{\alpha}(b-y)^{\beta}}\int_{x}^{a_0}\exp[A(a)(\omega_{\alpha}(t)-\omega_{\alpha}(x))-W^{-}_{A,\alpha}(t)-W^{-}_{A,\alpha}(x)]dt$$
$$\times\int_{b_0}^{y}\exp[B(b)(\omega_b^{\beta}(y)-\omega_b^{\beta}(s))+W^{-}_{b,\beta}(s)-W^{-}_{b,\beta}(y)]v(t,s)ds$$
$$\equiv (T^{\alpha,\beta}_{A(x),B(y)})^{-1}(g(x,y)),$$

for any $(x,y)\in D$, where $(T^{\alpha,\beta}_{A(x),B(y)})^{-1}$ is a known integral operator.

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References

[1] F. D. Gakhov, *Boundary value problems*, I. N. Sneddon, Ed. Oxford-New York-Paris; Reading, Mass.-London: Pergamon Press; Addison-Wesley Publishing Co., Inc., 1966.

[2] R. N., *Volterra type integral equations with boundary and interior fixed singular and supersingular kernels and their application*. Dushanbe: Tehran University, 2010.

[3] N. Rajabov, *An introduction to the theory of partial differential equations with super-singular coefficients*. Dushanbe: Tehran University, 1992.

[4] N. Rajabov, "On a Volterra integral equation," *Dokl. Math.*, vol. 65, no. 2, pp. 217–220, 2002.

[5] N. Rajabov, "About one class of Volterra type linear integral equations with an interior fixed singular or super-singular point," in *Topics in analysis and its applications*, ser. NATO Sci. Ser. II Math. Phys. Chem., vol. 147. Dordrecht: Kluwer Acad. Publ., 2004, pp. 317–326.

[6] N. Rajabov, *Volterra type integral equation with fixed boundary and interior singular and super–singular kernels*. Dushanbe: "Devashtich" Publishers, 2007.

[7] N. Rajabov, "About one class of second-order linear hyperbolic equations for which all of the boundary consist of singular lines," in *Further progress in analysis*. Hackensack, NJ: World Sci. Publ., 2009, pp. 356–367.
[8] N. Rajabov and L. Rajabova, "On the theory of a class of two-dimensional, non-model integral Volterra type equations with boundary supersingularities in the kernels," *Doklady Mathematics*, vol. 71, no. 1, pp. 111–114, 2005.
[9] N. Rajabov, M. Rontó, and L. Rajabova, "On some two dimensional Volterra type linear integral equations with super-singularity," *Math. Notes (Miskolc)*, vol. 4, no. 1, pp. 67–76, 2003.

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